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MATHEMATICAL

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Mixed type converse duality in multiobjective programming problems ☆

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Received 12 January 2004

Available online 29 January 2005

Submitted by B.S. Mordukhovich

Abstract

In this paper, we use the Fritz John necessary optimality conditions to establish some results on the mixed type converse duality for a class of multiobjective programming problems.

Keywords: Mixed type dual model; Converse duality; Multiobjective programming problem; Fritz John necessary condition

1. Introduction

Consider the following multiobjective nonlinear programming problem:

$$\text{(VP) Minimize } f(x)$$
$$\text{subject to} \quad g(x) \leqslant 0, \ x \in C, \tag{1}$$

☆ This research was partially supported by the National Natural Science Foundation of China (Grants 10171118 and 10471159), NCET and Natural Science Foundation of Chongqing.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.09.031

where $x \in \mathbb{R}^n$, f and g are twice continuousing differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively, C is an open subset of $\mathbb{R}^n$.

Denote by X the feasible set of (VP), $P = \{1, 2, \ldots, p\}$ and $M = \{1, 2, \ldots, m\}$. Let J_1 be a subset of M and $J_2 = M \setminus J_1$. Define $y_{J_k}^T g_{J_k} = \sum_{j \in J_k} y_j g_j$ for $k = 1, 2$, $\lambda \in \mathbb{R}^p$, $y \in \mathbb{R}^m$, and let e be the vector of $\mathbb{R}^p$ whose components are all ones.

In 1996, Xu [7] formulated the following mixed type dual for the problem (VP):

$$
\begin{aligned}
\text{(VD)} \quad & \text{Maximize } f(u) + y_{J_1}^T g_{J_1}(u) e \\
& \text{subject to} \quad \nabla f(u)^T \lambda + \nabla g(u)^T y = 0, && (2)\\
& \qquad y_{J_2}^T g_{J_2}(u) \geqslant 0, && (3)\\
& \qquad \lambda \geqslant 0, && (4)\\
& \qquad y \geqslant 0, && (5)\\
& \qquad \lambda^T e = 1, \; u \in C. && (6)
\end{aligned}
$$

Xu [7] studied weak and strong duality between (VP) and (VD) under generalized (F, ρ)-convexity conditions. Recently, Aghezzaf and Hachimi [1,2] obtained results on the weak and strong duality between (VP) and (VD) under different generalized convexity conditions. However, we note that in these papers, there is no discussion on converse duality between (VP) and (VD). More specifically, in [1,2], Aghezzaf and Hachimi only studied converse duality between (VD) and the following dual model, which is a special case of (VD):

$$
\begin{aligned}
(\text{VD}_1) \quad & \text{Maximize } f(u) \\
& \text{subject to} \quad \nabla f(u)^T \lambda + \nabla g(u)^T y = 0, \\
& \qquad y^T g(u) \geqslant 0, \\
& \qquad \lambda \geqslant 0, \\
& \qquad y \geqslant 0, \\
& \qquad \lambda^T e = 1, \; u \in C.
\end{aligned}
$$

In [5], Weir introduced following dual model, which is denoted by (VD_2). The converse duality theorem between (VP) and (VD_2) is then established for proper efficient solutions of a multiple objective programming:

$$
\begin{aligned}
(\text{VD}_2) \quad & \text{Maximize } f(u) + y^T g(u) e \\
& \text{subject to} \quad \nabla f(u)^T \lambda + \nabla g(u)^T y = 0, \\
& \qquad \lambda \geqslant 0, \\
& \qquad y \geqslant 0, \\
& \qquad \lambda^T e = 1, \; u \in C.
\end{aligned}
$$

In this paper, we use the weak duality obtained in [1] to establish a converse duality theorem between (VP) and (VD) for Pareto efficient solutions, the theorem is also shown to be valid for proper efficient solutions. Our results cover those obtained in [1,2] and [5] as special cases.

2. Main results

In this section, we will give the main results of the paper.

Theorem 1 (Converse duality). *Let* (x^*, y^*, λ) *be an efficient solution of* (VD) *at which*

(A1) *the* $n \times n$ *Hessian matrix* $\nabla^2[\lambda^T f(x^*) + y^{*T} g(x^*)]$ *is negative definite*; *and*
(A2) $\nabla y_{J_2}^T g_{J_2}(x^*) \neq 0$.

If the conditions of Theorem 4.1 *in* [1] *are satisfied*, *then* x^* *is an efficient solution of* (VP).

Proof. Since (x^*, y^*, λ) is an efficient solution of (VD), it follows from the generalized Fritz John necessary conditions [3] that there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^n$, $\theta \in \mathbb{R}$, $\eta \in \mathbb{R}^p$, $\xi \in \mathbb{R}^m$ and $p \in R$ such that

$$\begin{aligned}&-\alpha^T \nabla f(x^*) - (\alpha^T e)\nabla y_{J_1}^T g_{J_1}(x^*) + \beta^T\left[\nabla^2\lambda^T f(x^*) + \nabla^2\left(y^{*T} g(x^*)\right)\right]\\ &\quad - \theta\nabla y_{J_2}^T g_{J_2}(x^*) = 0,\end{aligned} \tag{7}$$

$$-(\alpha^T e) g_{J_1}(x^*) + \nabla g_{J_1}(x^*)^T \beta - \xi_{J_1} = 0, \tag{8}$$

$$-\theta g_{J_2}(x^*) + \nabla g_{J_2}(x^*)^T \beta - \xi_{J_2} = 0, \tag{9}$$

$$\nabla f(x^*)^T \beta + pe - \eta = 0, \tag{10}$$

$$\theta y_{J_2}^T g_{J_2}(x^*) = 0, \tag{11}$$

$$\xi^T y^* = 0, \tag{12}$$

$$\eta^T \lambda = 0, \tag{13}$$

$$p[\lambda^T e - 1] = 0,$$

$$(\alpha, \theta, \xi, \eta) \geqslant 0, \tag{14}$$

$$(\alpha, \beta, \theta, \xi, \eta, p) \neq 0. \tag{15}$$

Multiplying (10) by λ, it is clear from (13) that

$$\lambda^T\left(\nabla f(x^*)^T \beta + pe\right) = 0. \tag{16}$$

Multiplying (9) by y_{J_2} and using (11) and (12), we have

$$\nabla y_{J_2} g_{J_2}(x^*)^T \beta = 0. \tag{17}$$

Multiplying (7) by β, it follows from (2), (16) and (17) that

$$-\alpha^T\left[\nabla f(x^*)^T \beta + pe\right] + \beta^T\left[\nabla^2\lambda^T f(x^*) + \nabla^2\left(y^{*T} g(x^*)\right)\right]\beta = 0. \tag{18}$$

By (10), we obtain

$$-\alpha^T\left[\nabla f(x^*)^T \beta + pe\right] = -\alpha^T \eta. \tag{19}$$

Combining (18) and (19), we have

$$\alpha^T \eta = \beta^T\left[\nabla^2\lambda^T f(x^*) + \nabla^2\left(y^{*T} g(x^*)\right)\right]\beta. \tag{20}$$

Since $\nabla^2\lambda^T f(x^*) + \nabla^2 y^{*T} g(x^*)$ is, by (A1), negative definite, it is clear from (14) that $\beta = 0$.

Now we claim that $\alpha \neq 0$. Suppose it was false, then, by (7), we have

$$\theta \nabla y_{J_2}^T g_{J_2}(x^*) = 0. \tag{21}$$

By (A2), i.e., $\nabla y_{J_2}^T g_{J_2}(x^*) \neq 0$, it is clear from (21) that $\theta = 0$. By (8), $\xi_{J_1} = 0$, and by (9), $\xi_{J_2} = 0$, and by (6) and (13), it is clear from (10) that $p = 0$. Finally, by $\beta = 0$ and $p = 0$, it is clear from (10) that $\eta = 0$. That is, $(\alpha, \beta, \theta, \xi, \eta, p) = 0$, which is a contradiction to (15). Hence, $\alpha \neq 0$. Thus, $\alpha^T e > 0$. Since $\alpha^T e > 0$ and $\beta = 0$, it follows from (8) and (9) that

$$g(x^*) \leqslant 0.$$

Thus, x^* is a feasible solution of (VP). Therefore, by using the weak duality established of Theorem 4.1 of [1], we conclude that x^* is an efficient solution of (VP). $\square$

Remark 1. Note that if $J_1 = \emptyset$, then (VD) becomes Mond–Weir type dual model (VD_1). Thus, it is clear that the converse duality theorems obtained in Aghezzaf and Hachimi [1,2] are special cases of Theorem 1.

We now move on to study the converse duality for proper efficient solutions of a multi-objective programming under preinvexity conditions.

Lemma 1. *Let x and (u, y, λ) be, respectively, feasible solutions of* (VP) *and* (VD). *If f and g are preinvex (with respect to η) for all feasible solutions (x, u, λ, y), then*

$$\lambda^T\left[f(u) + y_{J_1}^T g_{j_1}(u)e\right] \leqslant \lambda^T f(x).$$

Proof.

$$\begin{aligned}
&\lambda^T\left\{f(x) - \left(f(u) + y_{J_1}^T g_{J_1}(u)e\right)\right\} \\
&\quad = \lambda^T(f(x) - f(u)) - y_{J_1}^T g_{J_1}(u), \quad \text{from (6)}, \\
&\quad \geqslant \eta^T(x,u)\nabla\lambda^T f(u) - y_{J_1}^T g_{J_1}(u), \quad \text{by preinvexity of } f, \\
&\quad = -\eta^T(x,u)\nabla y^T g(u) - y_{J_1}^T g_{J_1}(u), \quad \text{from (2)}, \\
&\quad \geqslant y^T g(u) - y^T g(x) - y_{J_1}^T g_{J_1}(u), \quad \text{by preinvexity of } g, \\
&\quad = y_{J_2}^T g_{J_2}(u) - y^T g(x) \\
&\quad \geqslant 0, \quad \text{since } g(x) \leqslant 0,\ y \geqslant 0 \text{ and (3)}.
\end{aligned}$$

Thus, $\lambda^T[f(u) + y_{J_1}^T g_{j_1}(u)e] \leqslant \lambda^T f(x)$. $\square$

Theorem 2 (Converse duality). *Let (x^*, y^*, λ) be a proper efficient solution of* (VD). *Assume that the following conditions are satisfied*:

(A1) *the $n \times n$ Hessian matrix $\nabla^2[\lambda^T f(x^*) + y^{*T} g(x^*)]$ is negative definite*;
(A2) $\nabla y_{J_2}^T g_{J_2}(x^*) \neq 0$ *and* $\lambda > 0$.

If the conditions of Lemma 1 *are satisfied, then x^* is a proper efficient solution of* (VP).

Proof. Since (x^*, y^*, λ) is a proper efficient solution of (VD), it is an efficient solution of (VD). By a similar argument given in the proof of Theorem 1, we can show that x^* is a feasible solution of (VP). By Lemma 1, we have

$$\lambda^T\left[f(x^*) + {y^*}_{J_1}^T g_{j_1}(x^*)e\right] \leqslant \lambda^T f(x), \quad \text{for any feasible solution } x \text{ of (VP)}. \tag{22}$$

Since $\alpha^T e > 0$, it follows from (8) and (12) that

$$y_{J_1}^T g_{J_1}(x^*) = 0. \tag{23}$$

Combing (22) and (23), we obtain

$$\lambda^T f(x^*) \leqslant \lambda^T f(x), \quad \text{for any feasible solution } x \text{ of (VP)}.$$

Thus, by Theorem 1 of [4], x^* is a properly efficient solution of (VP). $\square$

Remark 2. Note that preinvexity [6] is a generalized convexity. On the basis, if $J_2 = \emptyset$, then (VD) reduces to the dual model (VD_2). Thus, the results on the converse duality presented in Theorem 2 represent an improvement and extension of the main result obtained in Weir [5].

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